ON THE FLAVOUR INTERPRETATIONS OF STAGGERED FERMIONS

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The relationship between the flavour identifications of staggered fermions in configuration and in momentum space is clarified for interacting theories It is demonstrated that these identification schemes are identical in the continuum limit.

1. The staggered fermion formulation [1] has certain advantages over other methods. Firstly, since there is only one field component per site and per colour, there is a considerable advantage for numerical simulations. Secondly, from the theoretical point of view, it is also attractive because the formulation retains a continuous remnant of the chiral symmetry of the continuum theory.

However, the species-doubling problem is still present. The task of identifying these species as different flavours of quark is non-trivial, and has only recently been studied in the context of interacting theories [2–5]. The aim of this letter is to clarify these discussions.

In what follows we work on a four-dimensional, hypercubic, euclidean lattices of spacing a. Coordinates and fields are dimensionful. We use hermitean gamma matrices, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. The sixteen vectors A with $A_\mu = 0$ or 1 are frequently used. A basis for the Clifford algebra is then given by $\Gamma_A = \gamma_1^{A_1}\gamma_2^{A_2}\gamma_3^{A_3}\gamma_4^{A_4}$.

2. Starting from the free staggered fermion action with degenerate mass

$$S_{\mathrm{F}} = -a^4 \sum_x \left(\sum_\mu \alpha_\mu(x)\bar{\chi}(x) \frac{1}{2a}[\chi(x+a\hat{\mu}) - \chi(x-a\hat{\mu})] + m\bar{\chi}(x)\chi(x)\right),$$

$$x/a \in \mathbf{Z}^4 , \quad \alpha_\mu(x) = (-1)^{a^{-1}(x_1+\ldots+x_{\mu-1})} , \tag{1}$$

there are essentially two approaches to flavour identification. in configuration space [3,6], and in momentum space [2,4].

In configuration space, the first stage is to partition the lattice into hypercubes, thereby identifying sixteen one-component fields on a lattice of spacing $2a$ [3]

$$\chi_A(y) = \tfrac{1}{4}\chi(y + aA) , \quad \bar{\chi}_A(y) = \tfrac{1}{4}\bar{\chi}(y + aA) \quad y/2a \in \mathbf{Z}^4 , \tag{2}$$

Then, by means of a unitary transformation, quark fields with spinor (α) and flavour (a) indices are defined

$$\mathrm{q}^{\alpha a}(y) = \frac{1}{2}\sum_A \Gamma_A^{\alpha a}\chi_A(y) , \quad \bar{\mathrm{q}}^{\alpha a}(y) = \frac{1}{2}\sum_A \bar{\chi}_A(y)\Gamma_A^{*\alpha a} , \tag{3}$$

in terms of which the Fourier transformed action is

$$S_{\mathrm{F}} = -\int_{-\pi/2a}^{\pi/2a} \frac{\mathrm{d}^4p}{(2\pi)^4}\,\tilde{\bar{\mathrm{q}}}(p)\Big(\sum_\mu [(1/2a)\sin(2ap_\mu)\gamma_\mu \otimes \mathbf{1} - \{[1-\cos(2ap_\mu)]/2a\}\gamma_5 \otimes t_\mu t_5]$$

$$+ m\,\mathbf{1}\otimes\mathbf{1}\Big)\tilde{\mathrm{q}}(p)\,, \tag{4}$$

with $t_\mu = \gamma_\mu^*$, acting in flavour space. Here we have adopted the convention

$$\mathrm{q}(y) = \int_{-\pi/2a}^{\pi/2a} \frac{\mathrm{d}^4p}{(2\pi)^4}\exp(\mathrm{i}p\cdot y)\tilde{\mathrm{q}}(p)\,,\quad \bar{\mathrm{q}}(y) = \int_{-\pi/2a}^{\pi/2a} \frac{\mathrm{d}^4p}{(2\pi)^4}\exp(-\mathrm{i}p\cdot y)\tilde{\bar{\mathrm{q}}}(p)\,. \tag{5}$$

In momentum space, a very similar procedure can be adopted. After Fourier transforming eq. (1), using the same convention as in (5) with a replacing $2a$, we define sixteen one-component fields by partitioning the Brillouin zone [4]:

$$\tilde{\phi}_A(p) = \tilde{\chi}\,[p + (\pi/a)A]\,,\quad \tilde{\bar{\phi}}_A(p) = \tilde{\bar{\chi}}\,[p + (\pi/a)A]\,,\quad p \in [-\pi/2a\,,\pi/2a[\,. \tag{6}$$

In terms of these fields, (1) may be written

$$S_{\mathrm{F}} = -\int_{-\pi/2a}^{\pi/2a} \frac{\mathrm{d}^4p}{(2\pi)^4}\sum_{A,B}\tilde{\bar{\phi}}_A(p)\Big(\sum_\mu \frac{1}{a}\sin(ap_\mu)(\Gamma_\mu)_{AB} + m\delta_{AB}\Big)\tilde{\phi}_B(p)\,, \tag{7}$$

where, alternatively to the expressions of Γ_μ given in ref. [4] in terms of direct products of Pauli matrices, we can write.

$$(\Gamma_\mu)_{AB} = \tfrac{1}{64}\sum_{C,D}(-1)^{A\cdot C + B\cdot D}\,\mathrm{tr}(\Gamma_C^\dagger\gamma_\mu\Gamma_D)\,. \tag{8}$$

We now define quark fields with spinor (α) and flavour (a) indices via a unitary transformation V

$$\tilde{\psi}^{\alpha a}(p) = \sum_A V_A^{\alpha a}\tilde{\phi}_A(p)\,,\quad \tilde{\bar{\psi}}^{\alpha a}(p) = \sum_A \tilde{\bar{\phi}}_A(p)V_A^{*\alpha a}\,, \tag{9}$$

where

$$V_A^{\alpha a} = \frac{1}{8}\sum_B(-1)^{A\cdot B}\Gamma_B^{\alpha a}\,. \tag{10}$$

This differs from the transformation defined in ref. [4] by a rotation in flavour space; that is, their flavour matrices ξ_μ are related to t_μ by a unitary transformation. The action then takes the form

$$S_{\mathrm{F}} = -\int_{-\pi/2a}^{\pi/2a} \frac{\mathrm{d}^4p}{(2\pi)^4}\,\tilde{\bar{\psi}}(p)\Big(\sum_\mu \frac{1}{a}\sin(ap_\mu)\gamma_\mu \otimes \mathbf{1} + m\mathbf{1}\otimes\mathbf{1}\Big)\tilde{\psi}(p)\,. \tag{11}$$

Note that bot ϕ and ψ are completely non-local when expressed in terms of χ in configuration space.

The key difference between (4) and (11) is that the latter is diagonal in flavour space, whereas the former contains an explicit flavour mixing term of O(a) in the naive continuum limit. This is associated with the discontinuity at the Brillouin zone boundary exhibited by the propagator derived from (11), which indicates the non-locality of this interpretation in configuration space.

3. Gauge interactions are introduced in the usual manner

$$S_F = -a^4 \sum_x \Big(\sum_\mu \alpha_\mu(x) \frac{1}{2a} [\bar\chi(x)U_\mu(x)\chi(x+a\hat\mu) - \bar\chi(x+a\hat\mu)U_\mu^\dagger(x)\chi(x)] + m\bar\chi(x)\chi(x)\Big). \tag{12}$$

This interaction respects the symmetries (lattice, chiral and charge conjugation) of the free action, as discussed by Golterman and Smit [4]. These authors go on to show that the identification of the ψ-field indices can be made as in the free case, but only in the continuum limit. This is clear since ψ does not have a simple gauge transformation, owing to its non-locality in terms of χ. The same also applies to the q-fields, as will be explained below.

For this reason, it is tempting to define a gauge covariant quark field in configuration space, for example [3]

$$Q^{\alpha a}(y) = \frac{1}{2}\sum_A \Gamma_A^{\alpha a}\mathcal{U}_A(y)\chi_A(y), \quad \bar Q^{\alpha a}(y) = \frac{1}{2}\sum_A \bar\chi_A(y)\mathcal{U}_A^\dagger(y)\Gamma_A^{*\alpha a},$$

$$\mathcal{U}_A(y) = [U_1(y)]^{A_1}[U_2(y+aA_1)]^{A_2}[U_3(y+a(A_1+A_2))]^{A_3}[U_4(y+a(A_1+A_2+A_3))]^{A_4}, \tag{13}$$

Here, $\mathcal{U}_A(y)$ is a product of links from $y + aA$ to one specified corner of the hypercube, thus giving Q a simple (i.e. covariant) gauge transformation law. However, these covariant quark fields do have some disadvantages:

- Choosing one particular corner and one particular path as in (13) results in the Q-fields, and hadron fields constructed from them, losing most of the lattice symmetries. The implications of breaking the symmetry for the validity of the flavour identification, as well as for the restoration of the symmetry group in the continuum limit, are unclear. For example, there is no symmetry to prevent the occurrence of a linearly divergent term (i.e. an a^{-1} mass counterterm) in the Q-field propagator. Such a term would violate chiral symmetry, and introduce the fine-tuning problems familiar from Wilson and Kahler–Dirac fermions [7]. On the other hand, for the ψ-field propagator, symmetry does not allow this kind of term.
- Perturbation theory in terms of Q-fields is unpleasantly complicated, due to the appearance of $\mathcal{U}_A(y)$.
- Hadron fields defined via Q-fields contain more link variables than is necessary, since all the connection paths have to go through one particular corner of the hypercube. This leads to greater fluctuations and poorer statistics in a Monte Carlo simulation.

The q-fields, on the other hand, retain most of the lattice symmetries. We therefore propose to retain, even in the interacting case, the definition (3) for configuration space quark fields. As is the case with the ψ-fields, the flavour identification of the q-fields applies only in the continuum limit.

4. In fact, the definitions ψ and q coincide in the continuum limit. The general relationship between them may easily be derived. From eqs. (2), (3) and (6) we find

$$\tilde q^{\alpha a}(p) = \frac{1}{8}\sum_{A,B}\Gamma_B^{\alpha a}(-1)^{A\cdot B}\exp(\mathrm{i}ap\cdot B)\tilde\phi_A(p), \quad \tilde{\bar q}^{\alpha a}(p) = \frac{1}{8}\sum_{A,B}\tilde{\bar\phi}_A(p)\Gamma_B^{*\alpha a}(-1)^{A\cdot B}\exp(-\mathrm{i}ap\cdot B). \tag{14}$$

Hence, from (9) and (10)

$$\tilde\psi(p) = T(p)\tilde q(p), \quad \tilde{\bar\psi}(p) = \tilde{\bar q}(p)T^\dagger(p), \tag{15}$$

where $T(p)$ is a unitary transformation with the alternative forms

$$T(p)^{\alpha a,\beta b} = \frac{1}{4}\sum_A \Gamma_A^{\alpha a}\Gamma_A^{*\beta b}\exp(-\mathrm{i}ap\cdot A) = \exp\Big(\frac{\mathrm{i}a}{2}\sum_\mu p_\mu(-\mathbf{1}\otimes\mathbf{1} + \gamma_\mu\gamma_5\otimes t_\mu t_5)\Big)^{\alpha a,\beta b}. \tag{16}$$

For the free theory it is evident that $T(p)$ maps (11) into (4). This was the only property required of the transformation previously given in refs. [3,5], which differs by a momentum dependent phase factor from $T(p)$. However, this requirement does not determine the transformation uniquely but only up to a factor

$\exp(\mathrm{i}f\,\mathbf{1}\otimes t)$,

where f is an arbitrary real function of arbitrary parameters and t is any hermitian combination of the flavour matrices $\{\mathbf{1}, t_\mu, \mathrm{i}t_\mu t_\nu, \mathrm{i}t_\mu t_5, t_5\}$.

On the other hand, the transformation $T(p)$ is defined uniquely from the definitions of q and ψ, and thus the relationship (16) is valid in the general interacting case. Furthermore, since $T(p)$ depends only upon the physical momentum and a, and not on any bare quantities, it tends to the identity in the continuum limit. Hence, in this limit, we can identify the q-field indices with spin and flavour. In particular, it is evident that the self-energy of the q-fields is identical to that of the ψ-fields (calculated to one loop in ref. [4]), so that there is no linearly divergent term in the q-propagator.

This identification has been reinforced by the recent proof of the restoration of flavour symmetry in the continuum limit [5]. This proof can also be carried out using the transformation $T(p)$.

5 The transformations of the q-fields under the lattice symmetry can easily be derived either from those of the χ-fields [4,5] using (4), or from those of the ψ-fields [4] using (15) These laws are listed in the appendix of ref [5].

The fields ψ and q find their applications in different areas of analysis. The q-fields, being local (i.e. within one hypercube), are clearly amenable to use in numerical simulations, whereas weak coupling perturbation theory (necessary to make the connection to the continuum) is most easily carried out in terms of the ψ-fields [4]. At non-zero lattice spacing, both q and ψ are best regarded as rules for associating spin and flavour quantum numbers to suitably defined gauge invariant lattice meson and baryon operators [8,9].

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